

Compact $U(1)$ gauge theories in $2 + 1$ dimensions and the physics of low dimensional insulating materials

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Received: 13 October 2003 / Accepted: 22 October 2003 /

Published Online: 28 October 2003 – © Springer-Verlag / Società Italiana di Fisica 2003

Abstract. Compact abelian gauge theories in $d = 2 + 1$ dimensions arise often as an effective field-theoretic description of models of quantum insulators. In this paper we review some recent results about the compact abelian Higgs model in $d = 2 + 1$ in that context.

PACS. 11.15.Ha Lattice gauge theory – 11.10.Kk Field theories in dimensions other than four

1 Introduction

Effective abelian gauge theories of doped Mott insulators are very popular in the condensed matter literature. Despite the enormous amount of papers published over the last ten years, there are still many points of disagreement among researchers, mostly related to poorly understood issues [1, 2, 3, 4]. One key point is the interplay between the confinement/deconfinement properties of the many theories available, and spin-charge separation. It is usually believed that if deconfinement occurs, then spin-charge separation would also occur. However, it remains a controversial issue if deconfinement is possible for gauge fields coupled to matter with the fundamental charge. The problem here is that most $U(1)$ effective gauge theories are studied in $d = 2 + 1$ space-time dimensions. For this dimensionality $U(1)$ gauge fields are strongly confining. For instance, if we neglect the coupling to matter fields we obtain that pure compact Maxwell theory permanently confines electric charges in $d = 2 + 1$ dimensions [5]. This is in contrast with pure compact Maxwell theory in $d = 3 + 1$ dimensions where a deconfinement transition occurs [6]. A natural question to ask in this context is whether the coupling to matter fields is able to change the permanent confinement in $2 + 1$ dimensions. One example where the answer to this question is affirmative corresponds to the case where bosons are coupled to an abelian gauge field with integer charge $q > 1$ [7]. This is most easily understood by considering the lattice gauge theory version of such a model, whose action is given by

$$S = -\beta \sum_{i,\mu} [\cos(\nabla_\mu \theta_i - q A_{i\mu}) - 1] - \kappa \sum_{i,\mu,\nu} [\cos(F_{i\mu\nu}) - 1], \quad (1)$$

where we have $\nabla_\mu \theta_i \equiv \theta_{i+\hat{\mu}} - \theta_i$ and $F_{i\mu\nu} \equiv \nabla_\mu A_{i\nu} - \nabla_\nu A_{i\mu}$. The above action corresponds to the compact lat-

tice abelian Higgs (CLAH) model. When $\beta \rightarrow \infty$ the theory becomes a Z_q gauge theory which is known to have a deconfining transition in $2 + 1$ dimensions. For $\kappa \rightarrow \infty$ we obtain the XY model, which also undergoes a phase transition. The critical points corresponding to these two limiting cases are connected by a critical line separating the confining from the deconfining phase. There is no Coulomb phase in $2 + 1$ dimensions. The phase diagram and the critical properties of the action (1) were studied in detail using large scale Monte Carlo simulations in [8]. The CLAH model appears in many contexts in condensed matter physics. We shall cite only two important recent examples. In the $q = 2$ version it arises as an effective theory for the two-dimensional quantum Heisenberg antiferromagnet (QHA) [9]. There the gauge field originates from the Berry phase. Another recent example corresponds to the strongly correlated limit of a bosonic model considered by Senthil and Motrunich [10]. In this context the CLAH model describes a transition from an ordinary Mott insulating phase to a fractionalized insulating phase [10].

However, bosonic actions like the one in (1) are only part of the complete effective action associated with a doped Mott insulator. It usually contains also fermions coupled to a gauge field. [11, 12] The traditional route starts with a slave boson representation of the $t - J$ model where the projected electron operator is written as a composite particle, $c_{i\alpha} = b_i^\dagger f_{i\alpha}$, where b_i is an auxiliary boson and $f_{i\alpha}$ is an auxiliary fermion. The auxiliary fields obey the constraint $b_i^\dagger b_i + \sum_\alpha f_{i\alpha}^\dagger f_{i\alpha} = 1$ at each lattice site. In the resulting effective action the auxiliary fields interact through a compact gauge field. The most popular way of doing things attribute the charge of electron to the bosonic field. Thus, the fermion would have no charge and would only carry the spin degree of freedom. If the effective model undergoes a deconfinement transition fermions and bosons will have an independent dynamics and in this

way we can say the spin and charge are separated. Nayak [2] pointed out recently that things are not so simple because the assignment of the whole electron charge to the auxiliary boson is completely arbitrary. We could well say that the boson carries charge δe while the fermion carries charge $(1 - \delta)e$. The constraints of the theory, which are enforced by the coupling to the gauge field, would then ensure that the physics is unaffected by this arbitrary choice. This scenario implies that the auxiliary fields should not be associated with the physical charge and spin excitations of the theory. The auxiliary fields introduced this way are not part of the physical spectrum. This situation is reminiscent of the analysis made by Mudry and Fradkin some time ago. [13]

In this paper, we review some recent results on the model (1). We start discussing in the Sect. 2 the finite-size scaling (FSS) analysis of the model in the $q > 1$ case. We employ a new and very accurate method for extracting the critical exponents. In Sect. 3 we discuss the phase diagram, which is obtained for $q = 2, 3, 4, 5$. In Sect. 4 we discuss the $q = 1$ case, whose phase diagram is presently a matter of intensive debate [14, 15, 16].

2 The universality class of the deconfinement transition for $q > 1$

In 1981 Bhanot and Freedman [17] made a finite-size scaling (FSS) analysis of the CLAH model for $q = 2$ and $d = 2 + 1$. They obtained the phase diagram of the model but the critical exponents were not calculated. In order to determine the universality class of the model it is necessary to compute the critical exponents. The accurate determination of the critical behavior is not an easy task. In principle, on the basis of the results of Fradkin and Shenker [7], we might think that the universality class will be the same as of the Z_q spin model, except for the limit $\kappa \rightarrow \infty$ where the universality class is obviously that of the XY model. If we assume that this is indeed so, we obtain that the critical exponents for the $q = 2$ case have Ising values. However, we have recently shown through large scale Monte Carlo simulations that the situation is more complicated [8]. For example, for $q = 2$ we have obtained that the exponents are those of the Ising model in a large part of the phase diagram in the $\kappa\beta$ -plane, but there is a finite interval $\kappa_1 < \kappa < \kappa_2$ where the exponents vary continuously, before reaching XY values for $\kappa > \kappa_2$. Therefore, it seems that the model features a fixed line rather than a fixed point and belongs to a new universality class.

The FSS analysis of Bhanot and Freedman [17] relies on the second moment of the free energy, i.e., the specific heat. In this case it is well known that very large system sizes are needed in order to identify a genuine critical behavior to high accuracy. We have shown that much better results are obtained by performing a FSS analysis based on the third moment of the free energy. Such a procedure is not only more accurate: it allows also for an independent determination of the exponents ν and α , providing in this way a check of hyperscaling [8]. The reason for this lies in

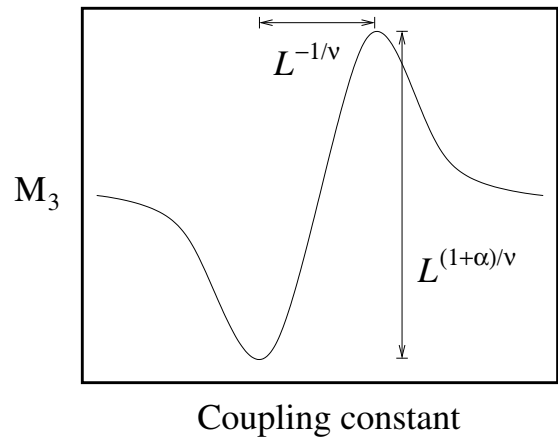


Fig. 1. Generic third moment M_3 , showing how FSS is used to extract α and ν

the double-peak structure of the third moment. The FSS in this case is such that the width between the negative and positive peak scales as $L^{-1/\nu}$, while the height of the positive peak with respect to the negative one scales as $L^{(1+\alpha)/\nu}$. This scaling behavior is shown schematically for a generic third moment M_3 in Fig. 2.

The critical exponents as a function of κ/β are shown in Fig. 2 for $q = 2$. It is clearly seen that the Ising critical behavior connects to the XY behavior through a regime of continuously varying critical exponents.

3 The phase diagram for $q > 1$

We have obtained the phase diagram for $q = 2, 3, 4, 5$. Among these values of q , only the $q = 3$ case exhibits a first-order phase transition up to some point where it changes to second-order. This behavior is consistent with the fact that the three-state Potts model exhibits a first-order transition in three dimensions. This model is dual to the Z_3 gauge theory. The phase diagram for $q = 3$ case therefore features a tricritical point. The tricritical point in the phase diagram is estimated as $(\beta_{\text{tri}}, \kappa_{\text{tri}}) = (1.23 \pm 0.03, 1.73 \pm 0.03)$, corresponding to a ratio $\kappa_{\text{tri}}/\beta_{\text{tri}} = 1.39 \pm 0.06$. At this point it is worth making the following remark. The non-compact version of the model (1) is known to exhibit a second-order phase transition for all values of q . As matter of fact, the value of q is not important in that case, and can be absorbed into a redefinition of the gauge field. The universality class corresponds to the so called “inverted” XY transition [18]. However, if the radial part of the scalar field is allowed to fluctuate, thus generalizing the non-compact version of (1), then it is possible to show using duality arguments that the resulting model has a tricritical point approximately at $\kappa_{\text{tri}}/\beta_{\text{tri}} \approx 0.64$ [19]. This estimate was confirmed recently by large scale Monte Carlo simulations [20]. In the case of the $q = 3$ CLAH model there is no need to consider the fluctuations of the radial component of the field to obtain a tricritical point. The origin of this tricritical point is completely different from the one of the non-compact mo-

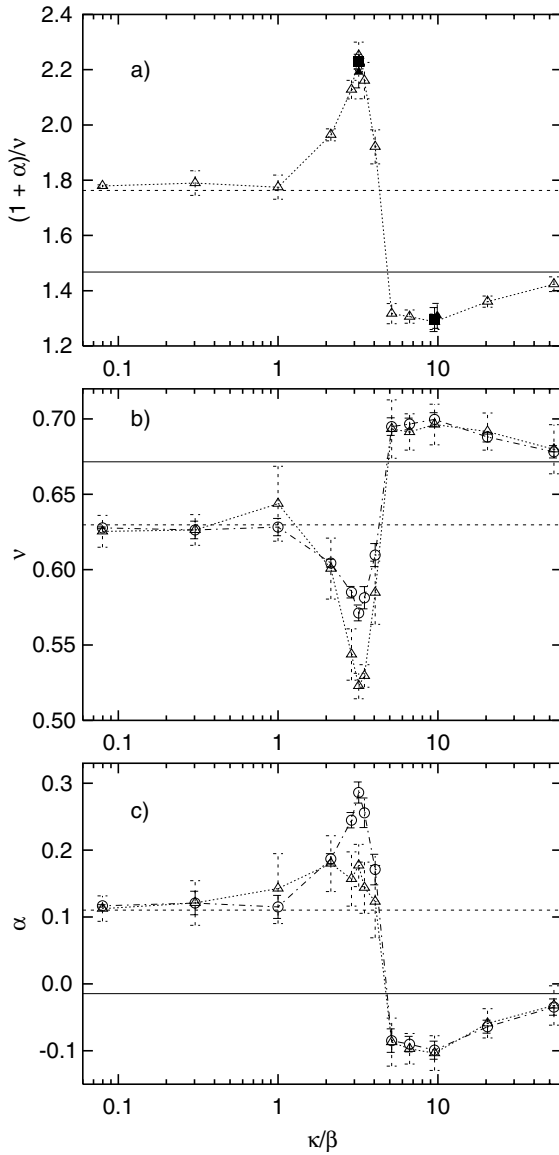


Fig. 2. a) $(1 + \alpha)/\nu$ from FSS finite-size of M_3 for (1) for $q = 2$. Note the variation relative to the Z_2 -limit 1.76 (dotted horizontal line) and the $U(1)$ -limit 1.467 (solid horizontal line). b) Same for the exponent ν , computed directly from M_3 (Δ) and combining results for $(1 + \alpha)/\nu$ with hyperscaling (\circ). c) α as computed directly from M_3 (Δ) and using results for $(1 + \alpha)/\nu$ with hyperscaling (\circ). The maximum and minimum in a) have been obtained by crossing the critical line along the trajectory $\beta(\kappa) = \beta_c + a(\kappa - \kappa_c)$ with $a = \infty$ (Δ), $a = 1$ (\blacksquare), and $a = -1$ (\blacktriangle) using $\beta_c = 0.665, \kappa_c = 2.125$ (max.), and $\beta_c = 0.525, \kappa_c = 5.0$ (min)

del. Anyway, it is interesting to note that the ratio $\kappa_{\text{tri}}/\beta_{\text{tri}}$ in the $q = 3$ CLAH model is more than twice larger than the corresponding ratio in the non-compact case.

In the context of the model discussed recently by Senthil and Motrunich [10], the phase diagram for the $q = 2$ case has the following physical meaning. The confined phase corresponds to gap to integer charge excitations and is therefore interpreted as an ordinary Mott insulating

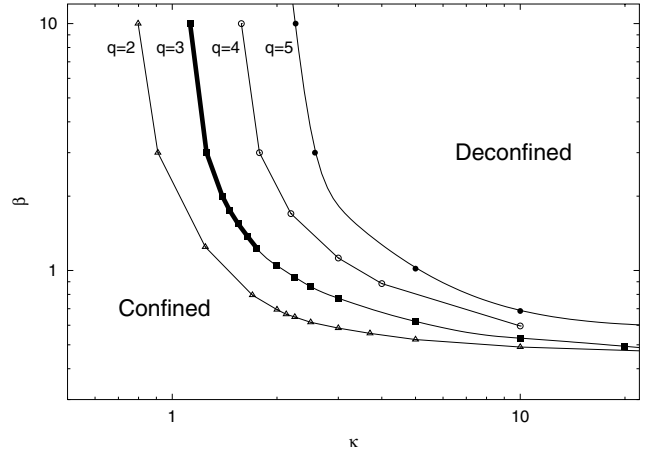


Fig. 3. The phase diagram for the $d = 3$ compact abelian Higgs model in three dimensions for $q = 2, 3, 4, 5$. All lines are critical for all values of κ except for the case $q = 3$, which is first order for $\kappa < \kappa_{\text{tri}}$ and second order otherwise. The thick solid portion of the $q = 3$ line is a first-order transition line. The critical line approaches the $3DXY$ value $\beta_c = 0.453$ as $\kappa \rightarrow \infty$ for all integer values of $q > 1$

phase. The deconfined phase, on the other hand, is the same as the Higgs phase. This gives a gap to half-integer charge excitations and is interpreted as a *fractionalized* insulator. The model can in principle be artificially realized with present day technology by building an array of Josephson junctions of a particular type [10].

4 The $q = 1$ case

The $q = 1$ case was generally believed to not exhibit any phase transition at all. The coupling to matter fields would not change the scenario of the $d = 2 + 1$ compact Maxwell theory, where it is known that no phase transition takes place [5]: the electric charges would never deconfine. A research initiated with the study of the QHA showed that things seem not to be so simple [11]. The main point is that by integrating out the matter fields we obtain to quadratic order in the gauge field fluctuations the effective action

$$S_{\text{eff}} \propto \int d^3x F_{\mu\nu} \frac{1}{\sqrt{-\partial^2}} F_{\mu\nu}. \quad (2)$$

Therefore, while in the absence of matter fields the monopoles interact through a $1/r$ potential, in the presence of matter fields this behavior changes to $-\ln r$. In the former case it is well known that upon dualizing the theory a sine-Gordon action is obtained. In the latter case, however, an *anomalous* sine-Gordon action arises [14]:

$$S = \frac{1}{8\pi^2 K} \int d^3x [\varphi(-\partial^2)^{3/2} \varphi - 2z \cos \varphi], \quad (3)$$

where $K = 1/g^2$, with g being the gauge coupling. Thus, we arrive at a behavior similar to the two-dimensional

case. By simply looking at the scaling behavior of the fugacity of the monopole gas, one would be lead to conclude that a Kosterlitz-Thouless (KT) phase transition happens in three space-time dimensions. *Note that this KT transition is at zero temperature and has nothing to do with other KT transitions obtained by dimensional reduction due to temperature effects.* However, it is not enough to study the scaling of just the fugacity. Since we are in $d = 2 + 1$ a more thorough analysis is needed in order to fully establish the KT behavior. To this end, it is necessary to also know the scaling behavior of the “stiffness” K , since the scaling of the fugacity and the scaling of the stiffness mutually influence each other in a subtle way.

For the usual sine-Gordon model in d dimensions we obtain the following coupled recursion relations for the fugacity and the stiffness [21]

$$\frac{dK^{-1}}{dl} = 4\pi^2 y^2 - (2 - d)K^{-1}, \tag{4}$$

$$\frac{dy}{dl} = [d - 2\pi^2 f(d)K] y, \tag{5}$$

where $f(d) = (d - 2)\Gamma[(d - 2)/2]/(4\pi)^{d/2}$. For $d = 2$ the above equations imply the existence of a line of fixed points characteristic of the KT transition. For $d = 3$, however, no fixed point is found and therefore no phase transition takes place. This is consistent with Polyakov’s result for the compact Maxwell theory in $d = 2 + 1$ [5]. The issue here is how the above recursion relations are modified in the case of the anomalous sine-Gordon model, (3). To investigate this in great generality we will consider screening in a Coulomb gas of monopoles with a propagator of the form $1/|p|^\sigma$ in $2 \leq d < 4$ dimensions with $\sigma > 0$. The particular case $d = \sigma = 3$ corresponds to our anomalous sine-Gordon theory. The bare potential is given by $U_0(r) = -4\pi^2 K V(r)$, where

$$V(r) = \frac{\Gamma\left(\frac{d-\sigma}{2}\right)}{2^\sigma \pi^{d/2} \Gamma(\sigma/2)} [(Ar)^{\sigma-d} - 1], \tag{6}$$

with A being an ultraviolet cutoff. Note that for $d = \sigma$ we obtain a potential $\propto \ln r$. The bare electric field is given by $E_0(r) = -4\pi^2 K A(d, \sigma) r^{\sigma-d-1} / r_0^{\sigma-d}$, where $r_0 \equiv 1/A$ and $A(d, \sigma) = (d - \sigma)\Gamma[(d - \sigma)/2]/[2^\sigma \pi^{d/2} \Gamma(\sigma/2)]$. The bare electric field is renormalized by the other dipoles which are treated as a dielectric medium. The *renormalized* electric field is then given by

$$E(r) = -\frac{4\pi^2 K A(d, \sigma) r^{\sigma-d-1}}{\varepsilon(r)}, \tag{7}$$

where $\varepsilon(r)$ is the scale-dependent dielectric constant of the medium. This problem can be solved self-consistently for a renormalized potential $U(r)$ derived from the above electric field [14]. In this way we obtain the recursion relations

$$\frac{dK^{-1}}{dl} = 4\pi^2 y^2 - (\sigma - d)K^{-1}, \tag{8}$$

$$\frac{dy}{dl} = [d - \eta_y - 2\pi^2 A(d, \sigma)K] y, \tag{9}$$

where the *anomalous dimension* of the fugacity is given by $\eta_y = (\sigma - 2)/2$. The above equations reduce to (4) and (5) when $\sigma = 2$. The case relevant to the $q = 1$ CLAH model is $\sigma = d = 3$. In this case (8) and (9) are very similar to the usual KT recursion relations, except for the presence of the anomalous scaling dimension of the fugacity, η_y which is nonzero in our case and given by $\eta_y = 1/2$. Thus, we obtain that the monopole gas undergoes a KT-like phase transition. This is in contrast with Polyakov’s compact Maxwell theory where the monopoles are always in the plasma phase. Using the usual duality arguments, we identify the plasma phase with the confined phase for the electric charges. The dielectric phase of the monopole gas is accordingly identified with the deconfined phase for the electric charges.

At this point an important remark is in order. The above screening analysis was made in real space, in the spirit of the original Kosterlitz and Thouless paper [22]. Indeed, we have considered a *local* scale dependent dielectric constant. Screening arguments in momentum space usually involve a dielectric constant which is local in momentum space, leading to an electric displacement vector of the form $\mathbf{D}(\mathbf{r}) = \int d^d r' \epsilon(\mathbf{r} - \mathbf{r}') \mathbf{E}(\mathbf{r}')$, while in our case we have simply $\mathbf{D}(\mathbf{r}) = \epsilon(\mathbf{r}) \mathbf{E}(\mathbf{r})$. In the present context this may be understood by using a classical electrostatic argument. Let us consider the potential (6) in the case $\sigma = d$, such that we have a $\ln r$ -potential in d dimensions. Due to Gauss’s theorem in d -dimensions, the field equation is given by [23] $\nabla \cdot (\mathbf{E} r^{2-d}) = S_d \rho(r)$, where S_d is the solid angle in d -dimensions. This can be cast in a more familiar form $\nabla \cdot \mathbf{D} = S_d r_0^{d-2} \rho(r)$ by introducing the “dielectric constant” $\varepsilon_0(r) = (r/r_0)^{2-d}$. Note that for $d > 2$ this dielectric constant vanishes at large distances. This is rather unusual since in classical electromagnetism it can be shown that the dielectric constant is always greater than one. However, such a situation corresponds precisely to the anti-screening mechanism discussed many years ago in theories of confinement [24]. Thus, already at this level, even before taking the dipoles into account, we can define a kind of scale-dependent dielectric constant which is local in real space.

5 Conclusions

In this paper we discussed the deconfinement transition in the CLAH model, (1), in $d = 2 + 1$. To each value of the charge q corresponds a distinct universality class, in contrast to the non-compact model. While a deconfined transition is expected for $q > 1$, the case $q = 1$ is still controversial [14, 15, 16] and needs further study. One feature that also deserves further study is the possible existence of a fixed line in the $q = 2$ case. The numerical results strongly suggest this possibility but multicritical fixed points are also possible.

Acknowledgements. FSN would like to thank Boris Pioline and Christoph Berger for the invitation to give the talk in the EPS HEP2003 conference, upon which this paper is based. We thank the Deutsche Forschungsgemeinschaft (DFG) and the Norwegian Research Council, Project No. 131520/432, for financial support.

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